The Painlevé Property, W Algebras and Toda Field Theories associated with Hyperbolic Kac-Moody Algebras

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Abstract

We show that the Painlevé test is useful not only for probing (non-) integrability but also for finding the values of spins of conserved currents (W currents) in Toda field theories (TFTs). In the case of the TFTs based on simple Lie algebras the locations of resonances are shown to give precisely the spins of conserved W currents. We apply this test to TFTs based on strictly hyperbolic Kac-Moody algebras and show that there exist no resonances other than that at $n = 2$, which corresponds to the energy-momentum tensor, indicating their non-integrability. We also check by direct calculation that there are no spin-3 nor -4 conserved currents for all the hyperbolic TFTs in agreement with the result of our Painlevé analysis.

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1 Introduction

Toda field theories define integrable field theories if they are associated with a Cartan matrix of a simple Lie algebra or a affine Kac-Moody algebra [1, 2, 3, 4, 5, 6, 7, 8]. They have broad application in mathematical and theoretical physics, and in particular have attracted particle physicists' interest in the various areas of researches [9, 10, 11, 12, 13, 14, 15].

A Toda field theory is governed by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{N} K_{ij}\partial_+\varphi_i\partial_-\varphi_j + \frac{1}{\beta}\sum_{i=1}^{N}\exp\left(\beta\sum_{j=1}^{N}K_{ij}\varphi_j\right), \tag{1}$$

where K_{ij} is a Cartan matrix of some Lie algebra $\boldsymbol{g}$. It is referred to as conformal and affine Toda field theory, respectively, depending on whether $\boldsymbol{g}$ is a simple Lie algebra or a affine Kac-Moody algebra. The former is a generalization of the Liouville theory. It is conformally invariant, admitting no soliton solutions. The latter is a relative of the sine-Gordon theory. The fields then become massive and the conformal invariance is lost. The addition of the extra potential term associated with the highest root can be regarded as a deformation of a conformal field theory [16]. The field equations have soliton solutions if the coupling constant is imaginary [13].

A more general class of Kac-Moody algebras can be defined in association with *generalized Cartan matrices* [17], which do not necessarily provide positive (semi-)definite root spaces. It is then interesting to ask what property the Toda theory will possess, if $\boldsymbol{g}$ is taken to be one in this class of Kac-Moody algebra. In this paper we will consider the case of *hyperbolic Kac-Moody algebra* (*hyperbolic Toda field theory, HTFT*), mainly focusing on the issue of its (non-)integrability.

One of the reasons why we are interested in the integrability of HTFTs concerns the existence of W currents [18]. Integrability of a Toda field theory reflects the existence of as many conserved currents as the number of degrees of freedom in general [19]. It was shown that one can reconstruct exact solutions of Toda field equations from solutions of Miura-type differential equations [20]. Since the conserved currents of spin-2 and higher associated with conformal Toda field theories are known to generate a W algebra [21, 22], one may expect that, if HTFTs are integrable, one may then obtain a new class of W algebras through their conserved currents which are supposed to exist. This will open up a new direction to extend a symmetry of conformal field theory and string theory. We hope our study will shed light from a physical point of view on the nature of hyperbolic Kac-Moody algebras, which has been scarcely understood as yet.

It is not easy in general to prove whether or not a given field theory is integrable (or non-integrable) in a rigorous sense. A practical method has been contrived for this aim by Weiss et. al. [23, 24], in which so-called 'the Painlevé property' was utilized to probe integrability of partial differential equations

(See [25] for a review.). It was demonstrated there that such analyses work very well in a number of integrable models. We call this 'experimental' test *the Painlevé test*. Applying this test to HTFTs, we show that the strictly hyperbolic Toda field theories (SHTFT) do not pass the test in the following sense: The minimal number of arbitrary functions that a generic solution of the Toda field equation possesses is smaller than that of ordinary integrable conformal Toda field theories if the solution is expanded around a singular manifold.

In the subsequent sections we will first 'rediscover' that, in the case of simple Lie algebras, resonances occur precisely at $n =$ exponents $+ 1$. This relation was reported earlier by Flaschka and Zeng [26]. We will give an alternative, direct proof to the key theorem for the relation. We will next show that a unique resonance occurs at $n = 2$ in the case of SHTFTs, suggesting that only a single conserved current (the energy-momentum tensor) exists for these theories. We will also check explicitly that there are indeed no spin-3 and -4 conserved currents for any HTFT, which supports the result of our Painlevé analysis.

The plan of this paper is as follows. In sect.2 we will give a brief review of the Painlevé test. We will devote sect.3 to the application of the test to the HTFTs. After the definition of the hyperbolic Kac-Moody algebra, we will prove that any Toda field theory associated with an invertible Cartan matrix is conformally invariant [27]. We will then see a beautiful relation between the resonances and the exponents of simple Lie algebras, and show further that any SHTFT does not pass the Painlevé test. In sect.4 we will check the non-existence of conserved currents of spin-3 and -4 for any HTFT. Finally we will summarize our results and future prospects in sect.5. Appendix A and B contain proofs of the Remark.2 and the Theorem, respectively.

2 Review of the Painlevé test

A system of ordinary differential equations are said to possess the Painlevé property if all its 'movable' singularities (singularities whose locations depend on the initial conditions) are pole singularity. The first observation on the relation between the Painlevé property and integrability was made by S.Kowalevskaya in 1889 in her work of rigid-body problems [28]. This property is named after Painlevé, who classified the second-order differential equations which possess such a property.

Weiss et. al. generalized the notion of the Painlevé property to partial differential equations of N complex variables $(z_1, \ldots, z_N)$. They assumed the form of the solutions as

$$u(z_1, \ldots, z_N) = \phi^{\alpha} \sum_{n=0}^{\infty} u_n \phi^n \tag{2}$$

in a neighborhood of a 'singular manifold'

$$\phi = \phi(z_1, \ldots, z_N) = 0, \tag{3}$$

and said that the partial differential equation has the Painlevé property when the expansion coefficients u_n consistently contain $2N-1$ arbitrary functions. They showed that a wide variety of known integrable equations, e.g. KdV, MKdV, Boussinesq, higher-order KdV and KP equations, enjoy this property [23, 24].

Example. Burger's equation

$$\dot{u} + uu' = \sigma u'' \tag{4}$$

($\dot{}$ and $'$ denote $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$, respectively.) is, substituting the expansion (2) into (4), reduced to the recursion relation of u_n:

$$\begin{aligned} & \dot{u}_{n-1} + (n-2)u_{n-1}\dot{\phi} + \sum_{m=0}^{n} u_{n-m}[u'_{m-1} + (m-1)\phi' u_m] \\ = \ & \sigma[u''_{n-2} + 2(n-2)u'_{n-1}\phi' + (n-2)u_{n-1}\phi'' + (n-1)(n-2)u_n(\phi')^2] \end{aligned} \tag{5}$$

Here α is determined to be -1 by the leading-order analysis. It turns out that (5) holds identically at $n=2$, and hence u_2 cannot be determined but is regarded as an arbitrary function. The values of n at which the recursion relation is identically satisfied (and hence there is a room for an arbitrary function) as above are called 'resonances' [29]. Integrability then requires $2N-1$ resonances.

It was shown by Yoshida [30] that the integrable Toda field theories ('generalized Toda lattice') based on simple Lie algebras are strongly characterized by their Painlevé property (See also [31]). Flaschka and Zeng were the first to show the correspondence between the locations of resonances and the exponents of the Lie algebra [26]. This relation was also known to Yoshida [32]. We found this relation independently; we will give another elementary proof of this, and generalize the analysis to HTFTs.

3 Painlevé test for SHTFTs

3.1 Hyperbolic Kac-Moody algebras

We now give a brief description of hyperbolic Kac-Moody algebras [17]. An $N \times N$ matrix K_{ij} is a generalized Cartan matrix if it satisfies (i) $K_{ij} \in \mathbf{Z}$, (ii) $K_{ii} = 2$, (iii) $K_{ij} \le 0$ ($i \ne j$) and (iv) $K_{ij} = 0$ if $K_{ji} = 0$. One can define a generalized Kac-Moody algebra by the generating relations

$$[h_i, h_j] = 0,\ [h_i, e_j] = K_{ij} e_j,\ [h_i, f_j] = -K_{ij} f_j,\ [e_i, f_j] = \delta_{ij} h_j \tag{6}$$

together with the Serre relations

$$(\mathrm{ad} e_i)^{1-K_{ij}}(e_j) = 0,\ (\mathrm{ad} f_i)^{1-K_{ij}}(f_j) = 0 \tag{7}$$

for $i \neq j$. A generalized Kac-Moody algebra is said hyperbolic Kac-Moody algebra (strictly hyperbolic Kac-Moody algebra, respectively) if the associated Dynkin diagram is of hyperbolic type (strictly hyperbolic type), i.e. if any deletion of nodes from the diagram leaves a direct sum of those of finite or affine type (finite type only). We will also use the same terminology for a Cartan matrix K.

Hyperbolic Kac-Moody algebras for $7 \leq \text{rank} \leq 10$ were first classified in [17]. The list of all the 136 hyperbolic Kac-Moody algebras for $3 \leq \text{rank} \leq 10$ was given in [33]. Together with all rank-2 generalized Kac-Moody algebras associated with Cartan matrices in the form [34]

$$K = \begin{bmatrix} 2 & -k \\ -l & 2 \end{bmatrix}, \quad k, l \in \mathbf{Z}, \quad kl > 4, \quad k, l > 0, \tag{8}$$

they exhaust all hyperbolic Kac-Moody algebras. These Kac-Moody algebras have root spaces with Lorentzian signature, and hence a HTFT contains a single ghost-like field.

It is only very recently that a few mathematicians have begun representation theoretic studies of generalized Kac-Moody algebras [35, 36] and that physicists have looked for applications of these algebras in particle physics (e.g. [37]). The most familiar example of hyperbolic Kac-Moody algebras will be E_{10}, which has the maximal allowed rank, in the context of string compactification (See [38, 39] for recent aspects on this subject.).

3.2 Conformal invariance of non-affine Toda field theories [27]

In the light-cone coordinates the Toda equation of motion is given by

$$\partial_+\partial_-\varphi_i = \exp\left(\beta \sum_{j=1}^{N} K_{ij}\varphi_j\right). \tag{9}$$

Under a conformal transformation

$$x^\pm \to \overline{x}^\pm(x^\pm) \tag{10}$$

the LHS of (9) changes as

$$\partial_+\partial_-\varphi_i \to \overline{\partial}_+\overline{\partial}_-\varphi_i = \frac{\partial x^+}{\partial \overline{x}^+}\frac{\partial x^-}{\partial \overline{x}^-}\partial_+\partial_-\varphi_i. \tag{11}$$

The invariance of the RHS of (9) then requires that

$$\varphi_i \to \overline{\varphi}_i = \varphi_i + \frac{\lambda_i}{\beta}\ln\left(\frac{\partial x^+}{\partial \overline{x}^+}\frac{\partial x^-}{\partial \overline{x}^-}\right) \tag{12}$$

for some λ_i such that

$$\sum_{j=1}^{N} K_{ij}\lambda_j = 1 \tag{13}$$

for any i. The solution is

$$\lambda_i = \left(K^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right)_i . \tag{14}$$

Hence one may find λ_i for any TFTs except affine TFTs. In other words, all but affine TFTs are conformally invariant. This fact implies the existence of a conserved chiral energy-momentum tensor in every non-affine TFTs.

In the case of the TFTs corresponding to simple Lie algebras, λ_i is related to 'half the sum of positive roots' (the Weyl vector). The above may be considered as a generalization (or a 'regularization') of this notion for general TFTs.

3.3 The Painlevé test

We now apply the Painlevé test to Toda field theories. We will closely follow ref.[30] (See Remark.1 below, however.). The Toda field equations are given by

$$\frac{\partial^2 \varphi_i}{\partial x \partial t} = -\exp\left(\sum_{j=1}^{N} K_{ij}\varphi_j \right) \quad (i = 1, \ldots, N). \tag{15}$$

We have omitted the coupling constant since it has no relevance here. Eq.(15) can be cast into an equivalent system of first order differential equations

$$\begin{aligned} \frac{\partial}{\partial t} A_i &= A_i \sum_{j=1}^{N} K_{ij} B_j \quad (i = 1, \ldots, N), \\ \frac{\partial}{\partial x} B_j &= -A_j = -\sum_{i=1}^{N} \delta_{ij} A_i \quad (i = 1, \ldots, N). \end{aligned} \tag{16}$$

Remark.1. Note that in ref.[30] a slightly different system of equations

$$\dot{A}^j = A^j \sum_{i=1}^{N} (D\eta)_j{}^i B_i, \quad B_i' = -\sum_{j=1}^{N} D_{ji} A^j \tag{17}$$

is adopted, where $\eta^{ij} = \delta^{ij}$ and D_{ij} is a matrix whose rows consist of simple roots (different from eq.(29)). Our choice (16) has an advantage in the hyperbolic case

in that the information on the signature of the root space is encoded only in the Cartan matrix K_{ij}, and hence we need not care about raising and lowering the indices by an indefinite metric.

Following the usual prescription of the Painlevé test, we assume that the solutions of (16) are single-valued around some singular manifold $\phi(x,t) = 0$. Substituting the expansion

$$A_i = \phi^{-n_A} \sum_{n=0}^{\infty} A_i^{(n)} \phi^n, \quad B_j = \phi^{-n_B} \sum_{n=0}^{\infty} B_j^{(n)} \phi^n \tag{18}$$

into (16), we have a recursion relation

$$T^{(n)} \vec{X}^{(n)} = \vec{b}^{(n)} \tag{19}$$

for the expansion coefficients $\vec{X}^n \equiv (A_1^{(n)}, \ldots, A_N^{(n)}, B_1^{(n)}, \ldots, B_N^{(n)})^T$ (T denotes the transpose.), where $T^{(n)}$ and $\vec{b}^{(n)}$ are given by

$$T^{(n)} = \begin{bmatrix} P^{(n)} & Q^{(n)} \\ R^{(n)} & S^{(n)} \end{bmatrix}, \qquad \vec{b}^{(n)} = \begin{bmatrix} b_i^{(n)} \\ b_{N+j}^{(n)} \end{bmatrix}, \tag{20}$$

$$\begin{aligned}
P_{ik}^{(n)} &= \left((n-2)\,\dot{\phi} - \sum_{j=1}^{N} K_{ij} B_j^{(0)} \right) \delta_{ik}, \\
Q_{il}^{(n)} &= -A_i^{(0)} K_{il}, \\
R_{jk}^{(n)} &= \delta_{jk}, \\
S_{jl}^{(n)} &= (n-1)\phi' \delta_{jl}, \\
b_i^{(n)} &= -\dot{A}_i^{(n-1)} + \sum_{m=1}^{n-1} A_i^{(n-m)} \sum_{j=1}^{N} K_{ij} B_j^{(m)}, \\
b_{N+j}^{(n)} &= -B_j^{(n-1)\prime}
\end{aligned} \tag{21}$$

$(i, j, k, l = 1, \ldots, N)$.

Remark.2. n_A and n_B can be shown to satisfy

$$n_A = n_B + 1. \tag{22}$$

The Laurent series (18) is called 'balance', and in particular is called 'lowest balance' if [40]

$$A_i^{(0)} \neq 0 \quad \text{for any } i = 1, \ldots, N. \tag{23}$$

It can be shown that for the TFTs based on either simple Lie algebras or strictly hyperbolic Kac-Moody algebras the only possibility is

$$n_A = 2, \quad n_B = 1, \tag{24}$$

while for general generalized Kac-Moody algebras

$$n_A = n_B + 1 \geq 3 \tag{25}$$

are allowed in general. However, (24) is only possible for lowest balances in the latter case as well, as far as K is invertible. The proof is given in Appendix A. In this paper we restrict ourselves to the lowest balances (and hence the case (24) only).

Evidently $\det T^{(n)}$ must vanish if n is a resonance, so let us calculate the determinant of $T^{(n)}$. Due to the assumption eq.(19) for $n = 0$ reads

$$B_i^{(0)} = -2 \dot{\phi} \sum_{j=1}^{N} (K^{-1})_{ij}, \quad A_i^{(0)} = -2 \dot{\phi} \phi' \sum_{j=1}^{N} (K^{-1})_{ij} \tag{26}$$

$(i = 1, \ldots, N)$. We write $T^{(n)}$ explicitly as

$$T^{(n)} = \left[\begin{array}{cccccc} n\dot{\phi} & & & -A_1^{(0)} K_{11} & \cdots & -A_1^{(0)} K_{1N} \\ & \ddots & 0 & \vdots & \cdots & \vdots \\ & 0 & \ddots & \vdots & \cdots & \vdots \\ & & n\dot{\phi} & -A_N^{(0)} K_{N1} & \cdots & -A_N^{(0)} K_{NN} \\ 1 & & & (n-1)\phi' & & \\ & \ddots & 0 & & \ddots & 0 \\ & 0 & \ddots & & 0 & \ddots \\ & & 1 & & & (n-1)\phi' \end{array} \right]. \tag{27}$$

It is easy to see that the determinant is given by

$$\det T^{(n)} = (\dot{\phi} \phi')^N \det[n(n-1) \cdot \mathbf{1} - 2DK], \tag{28}$$

where

$$D \equiv \left[\begin{array}{ccc} \sum_{j=1}^{N} (K^{-1})_{1j} & & \\ & \ddots & \\ & & \sum_{j=1}^{N} (K^{-1})_{Nj} \end{array} \right]. \tag{29}$$

Hence $\det T^{(n)} = 0$ is equivalent to the characteristic equation for the matrix $2DK$ of eigenvalues $n(n-1)$. In other words, the calculation of resonances is deduced to an eigenvalue problem.

We first give the results for the simple Lie algebras.

Proposition.1. *Set $\lambda = n(n-1)$, then $detT^{(n)}$ is given by $(\dot{\phi}\,\phi')^N \times$*

$$\begin{array}{rcl}
A_N & : & (\lambda-1\cdot 2)(\lambda-2\cdot 3)(\lambda-3\cdot 4)\cdots(\lambda-N(N+1)),\\
D_{2M} & : & (\lambda-1\cdot 2)(\lambda-3\cdot 4)(\lambda-5\cdot 6)\cdots(\lambda-(2M-3)(2M-2))\\
 & & \cdot(\lambda-(2M-1)2M)^2(\lambda-(2M+1)(2M+2))\\
 & & \cdots(\lambda-(4M-3)(4M-2)),\\
D_{2M+1} & : & (\lambda-1\cdot 2)(\lambda-3\cdot 4)(\lambda-5\cdot 6)\cdots\\
 & & \cdot(\lambda-(2M-1)2M)(\lambda-2M(2M+1))(\lambda-(2M+1)(2M+2))\\
 & & \cdots(\lambda-(4M-1)(4M)),\\
B_N(C_N) & : & (\lambda-1\cdot 2)(\lambda-3\cdot 4)(\lambda-5\cdot 6)\cdots(\lambda-(2N-1)2N),\\
E_6 & : & (\lambda-2)(\lambda-20)(\lambda-30)(\lambda-56)(\lambda-72)(\lambda-132),\\
E_7 & : & (\lambda-2)(\lambda-30)(\lambda-56)(\lambda-90)(\lambda-132)(\lambda-182)(\lambda-306),\\
E_8 & : & (\lambda-2)(\lambda-56)(\lambda-132)(\lambda-182)(\lambda-306)(\lambda-380)(\lambda-552)\\
 & & \cdot(\lambda-870),\\
F_4 & : & (\lambda-2)(\lambda-30)(\lambda-56)(\lambda-132),\\
G_2 & : & (\lambda-2)(\lambda-30).
\end{array} \tag{30}$$

The number appearing in each factor is always a product of two consecutive integers, the larger one of which corresponds to a (possible) resonance. Remarkably, we find that the (possible) resonances occur precisely at the values

$$n = \text{exponents} + 1 \tag{31}$$

for any Toda field theory based on a simple Lie algebra (Table 1). For simple Lie algebras with small rank we have checked the compatibility as well. We also notice that the resonances occur not $2N$ times, but only N (=rank) times. This fact, which may be seen as a discrepancy, has been known for some time, being interpreted as a limit where the missing $N-1$ arbitrary functions go to infinity [40].

This relation between the locations of the resonances and the exponents was first discovered by Flaschka and Zeng [26]. The Proposition.1 is a consequence of the following Theorem:

Theorem. *Let K be a Cartan matrix of a simple Lie algebra $\boldsymbol{g}$ of rank N, and D be a diagonal matrix defined in (29). Let $\{\chi_i|\ i=1,\ldots,N\}$ be the set of exponents of $\boldsymbol{g}$, then the set of eigenvalues of the matrix $2KD$ is given by $\{\chi_i(\chi_i+1)|\ i=1,\ldots,N\}$.*

In ref.[26] the above Theorem was proven by invoking the property of the Casimir of the principal sl(2) of $\boldsymbol{g}$ [41]. In Appendix B we will give an alternative, direct proof of the Theorem by means of induction for completeness.

Since the values of conserved W currents and the exponents of the Lie algebra are in one-to-one correspondence [22], the above result implies that the Painlevé test may tell us not only about its integrability, but also more detailed information about the spins of existing conserved currents. The Painlevé test can then be a powerful tool to search for new conserved W currents for Toda field theories based on generalized Kac-Moody algebras.

If this is true, then the Painlevé test should 'detect' the conserved energy-momentum tensor, which exists in any Toda field theory. In fact this is the case.

Proposition.2. The matrix DK and hence KD always has an eigenvalue 1.

Proof. The latter is given by

$$KD = \begin{bmatrix} K_{11} \sum_{j=1}^{N} (K^{-1})_{1j} & \cdots & K_{1N} \sum_{j=1}^{N} (K^{-1})_{Nj} \\ & \cdots & \\ K_{N1} \sum_{j=1}^{N} (K^{-1})_{1j} & \cdots & K_{NN} \sum_{j=1}^{N} (K^{-1})_{Nj} \end{bmatrix}. \tag{32}$$

Hence

$$\begin{aligned} KD \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} &= \begin{bmatrix} \sum_{k=1}^{N} K_{1k} \sum_{j=1}^{N} (K^{-1})_{kj} \\ \vdots \\ \sum_{k=1}^{N} K_{Nk} \sum_{j=1}^{N} (K^{-1})_{kj} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=1}^{N} \delta_{1j} \\ \vdots \\ \sum_{j=1}^{N} \delta_{Nj} \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \end{aligned} \tag{33}$$

This shows that $[1, \ldots, 1]$ is always an eigenvector of KD of eigenvalue 1, no matter what the Cartan matrix is (as far as it is invertible). q.e.d.

We have thus seen that there is always a resonance at $n = 2$ for any TFTs. We have also checked the compatibility of the recursion relation for rank-2 SHTFTs discussed below.

We will now apply the test to the HTFTs. In this paper we restrict our analysis to the SHTFTs (See sect.5, however.). The Cartan matrices of the strictly hyperbolic Kac-Moody algebras are classified into two classes. The first class consists of those for all rank-2 hyperbolic Kac-Moody algebras. They are of the form (8), and are infinite in number. The second class consists of those

associated with the eleven Dynkin diagrams listed up in the Table 2. In this case the rank is either three or four.

For the first class the determinant of $T^{(n)}$ is calculated as

$$\det T^{(n)} \doteq \phi\, \phi'(n+1)(n-2)\left[n(n-1) - 2\frac{(2+k)(2+l)}{4-kl}\right]. \tag{34}$$

Due to(8), k, l satisfy

$$\det K = 4 - kl < 0, \quad k, l > 0, \tag{35}$$

which means that

$$-2\frac{(2+k)(2+l)}{4-kl} > 0. \tag{36}$$

Hence the two solutions n of the equation $[\cdots]$ (in (34)) $= 0$ are either both positive or both negative. At the same time the sum of them must be 1. Clearly there exist no two integers that satisfy both requirements. This shows that the only positive integer solution of (34) $= 0$ is $n = 2$.

For the second class, on the other hand, we have explicitly checked for all the eleven cases that the matrix $2KD$ does not have any positive integer eigenvalue other than 2. The result is summarized in Table 3.

The Painlevé test thus suggests that the SHTFTs have no conserved currents except the energy-momentum tensor and that they are non-integrable.

4 Search for conserved currents in HTFTs

As we already discussed in Introduction, the conserved currents which are as many as the number of degrees of freedom in a conformal TFT are a consequence of their integrability (and vice versa). In this section we will study HTFTs from this point of view.

4.1 Spin 3

The equations of motion are given by (9). Due to the conformal invariance, we only consider the chiral (x^+-dependent) currents. The most general form of the spin-3 current is

$$W^{(3)} \equiv \sum_{i,j,k=1}^{N} a_{ijk}\partial_+\varphi_i\partial_+\varphi_j\partial_+\varphi_k + \sum_{i,j=1}^{N} b_{ij}\partial_+^2\varphi_i\partial_+\varphi_j + \sum_{i=1}^{N} c_i\partial_+^3\varphi_i. \tag{37}$$

Differentiating (37) by ∂_- and using the equation of motion (9), the current-conservation equation $\partial_- W^{(3)} = 0$ is reduced to

$$\begin{aligned}
&b_{ji} + \beta c_i K_{ij} = 0, \\
&(3a_{ijk} + \beta b_{ij}K_{ik} + \beta^2 c_i K_{ij}K_{ik}) + (j \leftrightarrow k) = 0, \\
&a_{ijk} = a_{jik}
\end{aligned} \tag{38}$$

for any i, j, k. There are two possible cases:
(i) $a_{ijk},\ b_{ij}\ (i, j, k = 1, 2)$ are written in terms of two independent coefficients c_1 and c_2.
(ii) $a_{ijk},\ b_{ij}\ (i, j, k = 1, 2)$ and c_1 are written in terms of c_2 only.
Note that the derivative of the energy-momentum tensor $\partial_+ T$ is always conserved, its coefficients satisfying (38). Since we are looking for spin-3 conserved currents other than $\partial_+ T$, we may set one of c_i to 0 from the outset. Therefore in the case (ii) all a_{ijk}, b_{ij} and c_i are zero, i.e. there are no other conserved currents, while in the case (i) the existence of another non-vanishing spin-3 conserved current is indicated.

For rank-2 Kac-Moody algebras with the Cartan matrix (8), the solution of (38) is given by

$$c_1 = \frac{l - kl}{k - k^2} c_2 \quad (k \neq 1,\ k > 0) \quad \text{or} \quad c_1 = \frac{l^2 - l}{1 - l} c_2 \quad (k = 1). \tag{39}$$

If $k \neq 1$ or $k = 1,\ l \neq 1$, c_i are related linearly, and hence there are no conserved currents. The case $k = l = 1$ corresponds to A_2 (su(3)).

4.2 Spin 4

We assume the form of the current as

$$\begin{aligned} W^{(4)} \quad \equiv \quad & \sum_{i,j,k,l=1}^{N} a_{ijkl} \partial_+ \varphi_i \partial_+ \varphi_j \partial_+ \varphi_k \partial_+ \varphi_l + \sum_{i,j,k=1}^{N} b_{ij} \partial_+^2 \varphi_i \partial_+ \varphi_j \partial_+ \varphi_k \\ & + \sum_{i,j=1}^{N} (c_{ij} \partial_+^2 \varphi_i \partial_+^2 \varphi_j + d_{ij} \partial_+^3 \varphi_i \partial_+ \varphi_j) + \sum_{i=1}^{N} e_i \partial_+^4 \varphi_i. \end{aligned} \tag{40}$$

$\partial_- W^{(4)} = 0$ is equivalent to

$$\begin{aligned} & d_{ji} + \beta e_i K_{ij} = 0, \\ & 2b_{jik} + 2\beta c_{ij} K_{ik} + \beta d_{ik} K_{ij} + 3\beta^2 e_i K_{ij} K_{ik} = 0, \\ & (4a_{ijkl} + \beta b_{ijk} K_{il} + \beta^2 d_{ij} K_{ik} K_{il}) + (j \leftrightarrow l) + (j \leftrightarrow k) = 0, \\ & a_{ijkl} = a_{jikl}, \\ & b_{jik} = b_{kij} \end{aligned} \tag{41}$$

for any i, j, k, l.

For TFTs based on rank-2 Kac-Moody algebras a calculation using MATHEMATICA shows that if

$$k > 0, \quad l > 0, \quad A \equiv 16 - 10k - 10l - 2kl - 3k^2 l - 3kl^2 \neq 0, \tag{42}$$

the equations (41) are solved as

$$
\begin{aligned}
c_{11} &= c_{22}\frac{l^2}{k^2} + e_2\left(\frac{4l - 3lk - l^2 + 2l^2k}{k^2}\right. \\
&\quad \left. + \frac{-10l - \frac{64l}{k^2} + \frac{56l}{k} + 6l^2 + \frac{88l^2}{k^2} - \frac{64l^2}{k} + 3kl^2 + 16l^3 - \frac{30l^3}{k^2} + \frac{2l^3}{k} - 6kl^3 - 6l^4 + \frac{9l^4}{k}}{A}\right) \\
c_{12} &= c_{22}\frac{l}{k} + e_2\left(\frac{7l - 3kl}{4} + \frac{3l}{2k}\right. \\
&\quad \left. - \frac{17l + \frac{8l}{k} + \frac{39kl}{2} + \frac{15k^2l}{2} + \frac{13l^2}{2} - \frac{5l^2}{k} - \frac{9kl^2}{2} - \frac{15k^2l^2}{4} + \frac{9k^3l^2}{4} + \frac{3l^3}{2} - \frac{9kl^3}{4} + \frac{9k^2l^3}{4}}{A}\right), \\
e_1 &= e_2\left(\frac{-10l + \frac{16l}{k} - 2l^2 - \frac{10l^2}{k} + 3kl^2 + 3l^3}{A}\right). \qquad (43)
\end{aligned}
$$

Hence the solution is expressed by two parameters c_{22} and e_2. Since $W^{(4)}$ includes two trivial conserved currents $\partial_+^2 T$ and $(T)^2$, two parameters can be set to zero from the beginning. This means that (43) corresponds to the cases with no conserved currents. Hence the necessary condition for such a current to exist is $A = 0$. This together with the condition (35) shows

$$
-4 < k + l < \frac{2}{3}. \qquad (44)
$$

It is clear that there are no k, l which satisfy both (35) and (44). This establishes the non-existence of non-trivial spin-4 conserved currents for HTFTs based on rank-2 Kac-Moody algebras.

For SHTFTs based on rank-3 and -4 Kac-Moody algebras we can also explicitly check that the solution of (41) is parameterized by only two parameters, and hence they have no conserved currents, either.

5 Summary and prospects

We have shown that the Painlevé test is useful not only for probing (non-)integrability but also for finding the values of spins of conserved currents in TFTs. The locations of resonances precisely give the spins of W currents for TFTs based on simple Lie algebras. We applied this test to SHTFTs, and showed that there exists no resonance except for the one at $n = 2$, which corresponds to the energy-momentum tensor, indicating the non-integrability. As a check, we have explicitly seen that there are no spin-3 nor -4 conserved currents for these theories.

One might think that the conformal invariance in two dimensions and the integrability are not compatible, since the former may lead to an infinite number of conserved charges. Our interpretation of this ‘discrepancy’ is as follows: Despite being infinite, the number of conserved charges for (S)HTFTs

may not be sufficient to be integrable. Namely, an (S)HTFT contains as many fields as the number of rank $N(\geq 2)$, hence there are '$N \times \infty$' independent modes, while a single conserved Virasoro current provides only '$1 \times \infty$' charges. We also recall an analogy in the relation between conformal field theories and integrable lattice models; the minimal series (i.e. $c < 1$) of the former correspond to the latter on criticality, but there is no such correspondence if $c(=$ the number of degrees of freedom) is larger than 1. In view of this, not all conformal field theory in two dimensions may necessarily be integrable if the number of fields> 1, unless other additional symmetries (e.g. W currents, Kac-Moody currents) exist in the system.

Finally, we will comment on general HTFTs. Although (24) is not the unique situation for general cases, there is no difficulty to perform the test for these theories in practice, starting from (23). We have also checked for all rank-3, -4 and -5 HTFTs that the resonance always occurs at $n = 2$ only. In view of this fact and our conserved-current analysis, we conjecture that all the HTFTs are non-integrable. One of the hints to clarify this point will be Ziglin's theorem, which was already used to show the integrability of some systems by Yoshida et. al. [42]. The cohomological analysis of Feigin and Frenkel may give us another suggestion on this problem [43]. It will also be interesting to establish the relation, if any, between the formal non-integer resonances and higher order Casimirs for generalized Kac-Moody algebras [44].

Appendix A

In this appendix we prove that (24) is the only possibility for TFTs based on simple Lie algebras or strictly hyperbolic Kac-Moody algebras. Let us assume $n_B \geq 2$. Substituting the expansion (18) in (16), we have

$$\sum_{m=0}^{n+n_B-1} A_i^{(n-m+n_B-1)} \sum_{j=1}^{N} K_{ij} B_j^{(m)} = \begin{cases} (n-n_A)\,\dot{\phi}\,A_i^{(n)} + \dot{A}_i^{(n-1)} & \text{if } n \geq 0, \\ 0 \quad \text{if } -n_B+1 \leq n \leq -1 \end{cases} \tag{45}$$

and

$$\sum_{n=0}^{\infty} \left\{ (n-n_B)\phi' B_j^{(n)} + B_j^{(n-1)\prime} \right\} \phi^{n-n_B-1} = -\sum_{n=0}^{\infty} A_j^{(n)} \phi^{n-n_A}, \tag{46}$$

where $A_j^{(-1)} \equiv B_j^{(-1)} \equiv 0$. We may, without loss of generality, assume that not all $A_i^{(0)}$ and not all $B_j^{(0)}$ are zero (because if so, we may then redefine $n_A \to n_A + 1$, etc.). (46) then means

$$n_B + 1 = n_A, \tag{47}$$

and hence

$$(n-n_B)\phi' B_j^{(n)} + B_j^{(n-1)\prime} = -A_j^{(n)}. \tag{48}$$

On the other hand, we find from (45) that

$$A_i^{(0)} \sum_{j=1}^{N} K_{ij} B_j^{(0)} = 0 \quad (i = 1, \ldots, N). \tag{49}$$

If

$$A_i^{(0)} \neq 0 \text{ for any } i = 1, \ldots, N \quad \text{(lowest balance)}, \tag{50}$$

then

$$B_1^{(0)} = B_2^{(0)} = \cdots = B_N^{(0)} = 0, \tag{51}$$

which contradicts the assumption (K is assumed to be invertible.). So let

$$A_1^{(0)} \neq 0, \ldots, A_P^{(0)} \neq 0, \quad A_{P+1}^{(0)} = \cdots = A_N^{(0)} = 0. \tag{52}$$

Substituting (52) into (48), we have

$$B_1^{(0)} \neq 0, \ldots, B_P^{(0)} \neq 0, \quad B_{P+1}^{(0)} = \cdots = B_N^{(0)} = 0. \tag{53}$$

Substituting (52)(53) into (49), we obtain

$$\sum_{j=1}^{P} K_{ij} B_j^{(0)} = 0 \quad (i = 1, \ldots, P). \tag{54}$$

If the $P \times P$ minor $\{K_{ij;\ i,j-1,\ldots,P}\}$ is invertible, then $B_1^{(0)} = \cdots = B_P^{(0)} = 0$, which contradicts (53). Hence it may not have its inverse. This cannot be satisfied by K_{ij} corresponding to simple Lie algebras or strictly hyperbolic Kac-Moody algebras, and the proof is thus completed.

Appendix B

In this Appendix we give a proof of the Theorem in sect.3. Here we show explicitly that the eigenvalues indeed coincide to the set of numbers known as the exponents, case by case. For A_N, B_N, C_N, D_N series we employ induction.

- A_N.

  The Cartan matrix and the associated D matrix are given by

$$\begin{aligned} K^{(N)} &= \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, & (55) \\ D^{(N)} &= \mathrm{Diag}\left[\frac{N \cdot 1}{2}, \frac{(N-1)\cdot 2}{2}, \ldots, \frac{(N-i+1)\cdot i}{2}, \ldots, \frac{1 \cdot N}{2}\right]. & (56) \end{aligned}$$

Induction. If $N = 1$, then $2K^{(1)}D^{(1)} = 2$, which trivially satisfies the Theorem. We next assume that $2K^{(N-1)}D^{(N-1)}$ has the spectrum $\{2, 6, 10, \ldots, N(N-1)\}$. The following lemma shows that the matrix $2K^{(N)}D^{(N)}$ also has the same eigenvalues:

Lemma A 1 *The matrix*

$$P^{(N)} \equiv \underbrace{\begin{bmatrix} N-1 & & & & & \\ 1 & N-2 & & & & \\ & 2 & \ddots & & & \\ & & \ddots & 2 & & \\ & & & N-2 & 1 \\ & & & & N-1 \end{bmatrix}}_{N-1} \left.\vphantom{\begin{bmatrix} 1\\1\\1\\1\\1\\1 \end{bmatrix}}\right\} N \tag{57}$$

is an 'intertwiner' of $2K^{(N)}D^{(N)}$ *and* $2K^{(N-1)}D^{(N-1)}$, *i.e. it satisfies*

$$2K^{(N)}D^{(N)} \cdot P^{(N)} = P^{(N)} \cdot 2K^{(N-1)}D^{(N-1)}. \tag{58}$$

Proof. Easy.

Therefore, to complete induction, we have only to prove that the extra eigenvalue of $2K^{(N)}D^{(N)}$ is $N(N+1)$. This can be shown by the following Lemma:

Lemma A 2

$$v = \left[1, -\frac{N}{2}, \frac{N(N-1)}{6}, \ldots, (-1)^{i+1}\frac{N!}{i!(N-i+1)!}, \ldots, (-1)^N\frac{N}{2}, (-1)^{N+1}\right]^T \tag{59}$$

is an eigenvector of $2K^{(N)}D^{(N)}$ *with the eigenvalue* $N(N+1)$.

Proof. Straightforward.

This completes the proof of the Theorem for the A_N type.

- B_N, C_N.

For these types one can prove the Theorem in the same way as we have done for A_N.

For B_N the Cartan matrix and the D matrix are

$$K^{(N)} = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{bmatrix}, \tag{60}$$

$$(D^{(N)})_{ij} = \begin{cases} i\left(N - \frac{i}{2} + \frac{1}{2}\right)\delta_{ij} & (1 \le i \le N-1), \\ \frac{1}{4}N(N+1)\delta_{Nj} & (i = N). \end{cases} \tag{61}$$

The assertion can be easily checked for $N = 2$. Making use of the following two Lemmas enables us to show, similarly by induction, that the matrix $2K^{(N)}D^{(N)}$ has the eigenvalues $\{1 \cdot 2,\ 3 \cdot 4,\ 5 \cdot 6, \dots,\ (2N-1) \cdot 2N\}$.

Lemma B 1 *The $N \times (N-1)$ matrix $P^{(N)}$, given by*

$$(P^{(N)})_{ij} = \begin{cases} (-1)^{i+j} \dfrac{2N(2N-2j-1)j!(2N-j-1)!}{i!(2N-i+1)!} \\ \qquad (1 \le i \le j \le N-1), \\ \dfrac{j}{2N-j} \quad (2 \le i = j+1 \le N), \\ 0 \quad \textit{(otherwise)}, \end{cases} \tag{62}$$

satisfies

$$2K^{(N)}D^{(N)} \cdot P^{(N)} = P^{(N)} \cdot 2K^{(N-1)}D^{(N-1)}. \tag{63}$$

Lemma B 2 *A column vector*

$$v_i^{(N)} = \frac{(-1)^i(2N-2i+1)}{i!(2N-i+1)!} \quad (i = 1, \dots, N) \tag{64}$$

is an eigenvector of $2K^{(N)}D^{(N)}$ of the eigenvalue $(2N-1)2N$.

Both Lemmas can be verified by straightforward calculations.

For C_N, the necessary informations are as follows:

$$K^{(N)} = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{bmatrix}, \tag{65}$$

$$(D^{(N)})_{ij} = \begin{cases} i\left(N - \frac{i}{2}\right)\delta_{ij} & (1 \le i \le N-1), \\ \frac{1}{2}N^2\delta_{Nj} & (i = N). \end{cases} \tag{66}$$

Lemma C 1

$$2K^{(N)}D^{(N)} \cdot P^{(N)} = P^{(N)} \cdot 2K^{(N-1)}D^{(N-1)}, \tag{67}$$

where

$$(P^{(N)})_{ij} = \begin{cases} (-1)^{i+j}\dfrac{2N(2N-1)(N-i)\cdot j!(2N-2-j)!}{(N-1)\cdot i!(2N-i)!} \\ \qquad\qquad\qquad (1 \le i \le j \le N-2), \\ \\ \dfrac{Nj}{(N-1)(2N-j-1)} \quad (2 \le i = j+1 \le N), \\ \\ (-1)^{N+i-1}\dfrac{(2N-1)(N-i)\cdot N!(N-2)!}{i!(2N-i)!} \\ \qquad\qquad\qquad (j = N-1, 1 \le i \le N-1), \\ \\ 0 \quad \textit{(otherwise)}. \end{cases} \tag{68}$$

Lemma C 2 *A column vector*

$$v_i^{(N)} = \frac{(-1)^i}{i!(2N-i)!} \quad (i = 1, \ldots, N) \tag{69}$$

is an eigenvector of $2K^{(N)}D^{(N)}$ *of the eigenvalue* $(2N-1)2N$*.*

The proof is completely parallel, and we leave it to the reader.

- D_N.

The Cartan matrix and the D matrix of the D_N type Lie algebra are

$$K^{(N)} = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & -1 \\ & & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 2 \end{bmatrix}, \tag{70}$$

$$(D^{(N)})_{ij} = \begin{cases} i\left(N - \frac{i}{2} - \frac{1}{2}\right)\delta_{ij} & (1 \le i \le N-2), \\ \frac{1}{4}N(N-1)\delta_{ij} & (i = N-1, N). \end{cases} \tag{71}$$

What we have to show is that the matrix $2X^{(N)} \equiv 2K^{(N)}D^{(N)}$ has eigenvalues $\{1\cdot 2,\ 3\cdot 4, \ldots,\ (2N-3)(2N-2);\ (N-1)N\}$ for any $N = 4, 5, \ldots$. Due to the 'middle' eigenvalue $(N-1)N$, we need some preparation before applying induction to this case.

Let us consider a symmetric $Y^{(N)} \equiv (D^{(N)})^{\frac{1}{2}} K^{(N)} (D^{(N)})^{\frac{1}{2}}$, which has the same set of eigenvalues as $X^{(N)}$ (Here we have already used the assumption of induction in anticipating reality of the square root of $D^{(N)}$). It is easy to see that

Lemma D 1 *$u^{(N)} \equiv [0, \ldots, 0, -1, 1]^T$ is an eigenvector of $2Y^{(N)}$ of the eigenvalue $N(N-1)$.*

This shows that $2X^{(N)}$ also has the eigenvalue $N(N-1)$. Since $Y^{(N)}$ is symmetric,

$$\frac{1}{2} u^{(N)} (u^{(N)})^T \tag{72}$$

is a projection operator to the vector space spanned by $u^{(N)}$. Hence, due to the assumption, the matrices

$$2Y^{(N-1)'} \equiv 2Y^{(N-1)} - \frac{(N-1)(N-2)}{2} u^{(N-1)} (u^{(N-1)})^T \tag{73}$$

and

$$\begin{aligned} 2X^{(N-1)'} &\equiv 2X^{(N-1)} - (D^{(N)})^{-\frac{1}{2}} \frac{(N-1)(N-2)}{2} u^{(N-1)} (u^{(N-1)})^T (D^{(N)})^{\frac{1}{2}} \\ &= 2X^{(N-1)} - \frac{(N-1)(N-2)}{2} u^{(N-1)} (u^{(N-1)})^T \end{aligned} \tag{74}$$

have eigenvalues $\{0,\ 1 \cdot 2, \ldots,\ (2N-5)(2N-4)\}$.

It now suffices to show that $2X^{(N)'}$ also possesses the same spectrum except one extra eigenvalue $(2N-3)(2N-2)$, as well as that, for the D_4 case, $2X^{(4)'}$ has eigenvalues $\{0, 2, 12, 30\}$. The latter can be done easily. To show the former, we can take the same steps as the previous proofs for other types of Lie algebras.

Lemma D 2 *The $N \times (N-1)$ matrix $P^{(N)}$, given by*

$$(P^{(N)})_{ij} = \begin{cases} (-1)^{i+j} \dfrac{2N(N-1)(2N-2j-3)j!(2N-j-3)!}{(N-2)i!(2N-i-1)!} & (1 \le i \le j \le N-3), \\ (-1)^{N+i} \dfrac{N((N-1)!)^2}{(N-2)i!(2N-i-1)!} & (j = N-2, N-1; \quad 1 \le i \le N-2), \\ \dfrac{jN}{(N-2)(2N-j-2)} & (2 \le i = j+1 \le N-2), \\ \delta_{ij} & (i = N-1, N; \quad j = N-2, N-1), \\ 0 & \textit{(otherwise)} \end{cases} \tag{75}$$

satisfies

$$2X^{(N)'} \cdot P^{(N)} = P^{(N)} \cdot 2X^{(N-1)'}. \tag{76}$$

One can prove the above by a straightforward calculation in the same manner.

The following Lemma completes the proof of the D_N case:

Lemma D 3 $2X^{(N)}$ *has an eigenvalue* $(2N-3)(2N-2)$.

Proof. It is easy to verify that the column vector

$$v_i^{(N)} \equiv \begin{cases} \dfrac{(-1)^i(2N-2i-1)}{i!(2N-i-1)!} & (1 \leq i \leq N-2), \\ \\ \dfrac{(-1)^{N+1}}{(N-1)!N!} & (i = N-1, N) \end{cases} \tag{77}$$

satisfies

$$2X^{(N)}v^{(N)} = (2N-3)(2N-2)v^{(N)}. \tag{78}$$

q.e.d.

- Exceptional types. For the exceptional types of simple Lie algebras one can explicitly establish the validity of the Theorem. We will list the eigenvectors for these cases for completeness:

  - E_6.

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \tag{79}$$

$$D = \mathrm{Diag}\,[8, 15, 21, 11, 15, 8], \tag{80}$$

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -2/5 & 0 & 0 & 2/5 & 1 \\ 1 & 1/15 & -1/3 & -1 & 1/15 & 1 \\ 1 & -4/5 & -16/35 & 8/5 & -4/5 & 1 \\ -1 & 4/3 & 0 & 0 & -4/3 & 1 \\ 1 & -10/3 & 16/3 & -28/11 & -10/3 & 1 \end{bmatrix}^T, \tag{81}$$

$$P^{-1}2KDP = \mathrm{Diag}\,[2, 20, 30, 56, 72, 132]. \tag{82}$$

– E_7.

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \tag{83}$$

$$D = \left[\frac{27}{2}, 26, \frac{75}{2}, 48, \frac{49}{2}, 33, 17\right], \tag{84}$$

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -65/33 & -10/11 & -31/165 & 17/66 & 4/11 & 19/33 & 1 \\ 13/22 & -1/44 & -5/22 & -37/176 & -37/77 & 2/11 & 1 \\ -65/9 & 5 & 53/15 & -1/2 & -6 & -1/3 & 1 \\ -5/12 & 5/8 & -1/12 & -17/48 & 1 & -32/33 & 1 \\ 10/11 & -320/143 & 2 & 6/11 & -48/77 & -19/11 & 1 \\ -34/99 & 238/143 & -170/39 & 68/11 & -408/143 & -119/33 & 1 \end{bmatrix}^T, \tag{85}$$

$$P^{-1} 2KDP = \mathrm{Diag}\,[2, 30, 56, 90, 132, 182, 306]. \tag{86}$$

– E_8.

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \tag{87}$$

$$D = \mathrm{Diag}\,[29, 57, 84, 110, 135, 68, 91, 46], \tag{88}$$

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -38/13 & -20/13 & -103/182 & 6/13 & 15/13 & 15/26 & 64/91 & 1 \\ 228/245 & -32/245 & -97/245 & -807/2695 & -2/21 & -9/49 & 2/7 & 1 \\ -19/15 & 11/15 & 67/105 & 1/15 & -1/3 & -1 & 1/91 & 1 \\ -33/13 & 55/13 & -99/91 & -213/91 & -397/819 & 5955/1547 & -61/91 & 1 \\ 4/13 & -176/247 & 7/13 & 17/65 & -18/65 & 9/13 & -14/13 & 1 \\ -874/1911 & 10028/5733 & -5267/1638 & 4301/1911 & 874/819 & -1311/1274 & -184/91 & 1 \\ 437/4901 & -23/39 & 1127/5507 & -4301/845 & 437/65 & -513/169 & -49/13 & 1 \end{bmatrix}^T, \tag{89}$$

$$P^{-1} 2KDP = \mathrm{Diag}\,[2, 56, 132, 182, 306, 380, 552, 870]. \tag{90}$$

– F_4.

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \tag{91}$$

$$D = \mathrm{Diag}\,[11, 21, 15, 8]\,, \tag{92}$$

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1/3 & 1/15 & 1 \\ 8/5 & -16/35 & -4/5 & 1 \\ -28/11 & 16/3 & -10/3 & 1 \end{bmatrix}^T, \tag{93}$$

$$P^{-1} 2KDP = \mathrm{Diag}\,[2, 30, 56, 132]\,. \tag{94}$$

– G_2.

$$K = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}, \tag{95}$$

$$D = \mathrm{Diag}\,[5, 3]\,, \tag{96}$$

$$P = \begin{bmatrix} 1 & 1 \\ -9/5 & 1 \end{bmatrix}^T, \tag{97}$$

$$P^{-1} 2KDP = \mathrm{Diag}\,[2, 30]\,. \tag{98}$$

We have thus proven the Theorem for all the types of simple Lie algebras.

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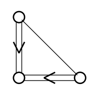
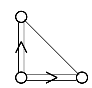
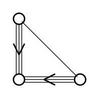
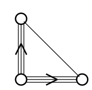
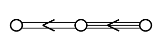
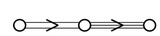
Table 1: Exponents for simple Lie algebras.

$\boldsymbol{g}$	Exponents
A_N	$1, 2, 3, \ldots, N$
B_N (C_N)	$1, 3, 5, \ldots, 2N-1$
D_N	$1, 3, 5 \ldots, 2N-3$ and $N-1$
E_6	$1, 4, 5, 7, 8, 11$
E_7	$1, 5, 7, 9, 11, 13, 17$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	$1, 5, 7, 11$
G_2	$1, 5$

Table 3: Eigenvalues of $2KD$ for strictly hyperbolic Kac-Moody algebras

Rank	Number	Eigenvalues	Number	Eigenvalues
3	1	$2, -12, -12$	1d	$2, -9, -15$
	2	$2, -6, -24$	2d	$2, -10, -20$
	3	$2, -\frac{15}{2}, -\frac{15}{2}$	3d	$2, -\frac{9}{2}, -\frac{21}{2}$
	4	$2, -12, -42$	4d	$2, \frac{-54\pm 2\sqrt{113}}{2}$
	5	$2, \frac{-58\pm 2\sqrt{281}}{2}$	5d	$2, \frac{-58\pm 2\sqrt{181}}{2}$
4	1	$2, -12, \frac{-30\pm 2\sqrt{57}}{2}$		

Table 2: Dynkin diagrams for strictly hyperbolic Kac-Moody algebras.

Rank	Number	$\boldsymbol{g}$	Number	$\boldsymbol{g}$ dual
3	1		1d	
	2		2d	
	3		3d	
	4		4d	
	5		5d	
4	1